Matematisk seminar
Universitetet i Oslo

Nr. 4
Oktober 1968.

MEASURES WHICH ACT ALMOST INVARIANTLY

by

R. Larsen.

MEASURES WHICH ACT ALMOST INVARIANTLY

R. Larsen.

1. Measures which act invariantly. Consider a locally compact abelian topological (LCA) group G and a complex valued regular Borel measure μ defined on G which is finite on compact sets. If $s \in G$ then we denote by $T_s\mu$ that measure which is defined by the equation $T_s\mu(E) = \mu(Es)$. A linear subspace X of $C_o(G)$, the sup norm Banach algebra of all complex valued continuous functions on G which vanish at infinity, is said to be translation invariant if $h \in X$ implies $T_s h \in X$ for all $s \in G$ where $T_s h(t) = h(t\,s)$. Suppose $X \in C_o(G)$ is such a subspace and furthermore that μ is a measure on G for which :

i) $\int_G |h(t)|\,d|\mu|(t) < \infty \qquad (h \in X),$

and

ii) $\int_G T_{s^{-1}} h(t)\,d\mu(t) = \int_G h(t)\,dT_s\mu(t) = \int_G h(t)\,d\mu(t) \qquad (h \in X,\ s \in G).$

If this is the case we say that μ acts invariantly on X.

Clearly if μ is a constant multiple of Haar measure m on G and $X \in C_c(G)$, the subspace of $C_o(G)$ consisting of functions with compact support, then μ acts invariantly on X . However assertions in the converse direction are not generally valid. Indeed there exist positive measures μ distinct from Haar measure and translation invariant subspaces X of $C_c(G)$ which are dense in $C_o(G)$ such that μ acts invariantly on X . For example, let $G = R$, the additive group of the real line , $d\mu(t) = (2 + \sin t)\,dm(t)$ and suppose X consists of all $h \in C_c(R)$ such that $\hat{h}(1) = \hat{h}(-1) = 0$, where $\hat{h}$ denotes the usual Fourier transformation of h. Then X is a translation invariant subspace of $C_c(R)$ which is dense in $C_o(R)$ and for each $s \in G$ and $h \in X$ we have

$$\begin{aligned}
\int_R T_{s^{-1}}h(t)\,d\mu(t) &= 2\int_R T_{s^{-1}}h(t)\,dm(t) + \frac{1}{2i}\int_R T_{s^{-1}}h(t)\left(e^{it}-e^{-it}\right)dm(t) \\
&= 2\int_R h(t)\,dm(t) + \frac{1}{2i}\left(e^{is}\,\hat{h}(-1) - e^{-is}\,\hat{h}(1)\right) \\
&= 2\int_R h(t)\,dm(t) \\
&= 2\int_R h(t)\,dm(t) + \frac{1}{2i}\left(\hat{h}(-1) - \hat{h}(1)\right) \\
&\int_R h(t)\,d\mu(t)\;.
\end{aligned}$$

It should be noted for future reference that X is <u>not</u> a subalgebra of $C_o(R)$. To see this it is sufficient to observe that $h(t) = \max(2\pi - |t|\;, 0)$ belongs to X but $\hat{h}^2(1) = 8\pi \neq 0$.

We are thus led to a consideration of the following questions. If μ acts invariantly on X then:

I) What can be said about μ? In particular, under what conditions is μ a constant multiple of Haar measure m?

II) What can be said about the linear (not necessarily continuous) functional F_e defined on X by

$$F_e(h) = \int_G h(t)\,d\mu(t) \qquad (h \in X)\;?$$

Such questions are of some importance since a frequent argument in harmonic analysis is to construct a measure μ which acts invariantly on a certain subspace X of $C_o(G)$ and then conclude that μ is a multiple of Haar measure. For instance the proofs of the Fourier inversion theorem in (6, 7, 8) depend on such an argument.

The problem we have just described has been studied by Jerison and Rudin (2) . We mention two of their results.

THEOREM 1. Suppose X is a translation invariant subalgebra of $C_o(G)$, which is dense in $C_o(G)$. If μ acts invariantly on X then μ is a constant multiple of Haar measure on G .

THEOREM 2. Suppose X is a translation invariant subspace of $C_c(G)$. If μ acts invariantly on X and there exists some $g \in X$ such that $\int_G g(t)dm(t) \neq 0$ then there exists a consant c such that

$$F_e(h) = c\int_G h(t)dm(t) \qquad (h \in X) .$$

The constant $c = \int_G g(t)d\mu(t) \ / \int_G g(t)dm(t)$.

REMARKS. a) The example previously discussed shows that the assumption that X is an algebra can not be dropped.

b) This example also shows that even when F_e can be considered as integration with respect to a constant multiple of Haar measure that μ need not be such a measure.

c) Without the assumption in Theorem 2 that $\int_G g(t)dm(t) \neq 0$ for some $g \in X$ the theorem may fail. For instance let X be the subspace generated by all the translates of even functions h in $C_c(R)$ for which $\int_R h(t)dm(t) = 0$ and set $d\mu(t) = t^2dm(t)$. Then μ acts invariantly on X since if $h \in X$ is even then

$$\int_R T_s h(t)d\mu(t) = \int_R h(t)(t^2 - 2st + s^2)dm(t)$$
$$= \int_R h(t)t^2dm(t) .$$

Moreover $\int_R h(t)dm(t) = 0$ for all $h \in X$.

However the function

$$f(t) = \begin{cases} 0 & , t > 6 \\ \frac{1}{3}t - 2 & , 3 < t \leq 6 \\ 2 - t & , 0 \leq t \leq 3 \\ f(-t) & , t < 0 \end{cases}$$

belongs to X and $\int_R f(t)d\mu(t) < 0$. Hence the conclusion of Theorem 2 is not valid.

2. Measures which act almost invariantly. In the examples discussed in the preceding section and in (2) all of the measures which act invariantly, though they may not be constant multiples of Haar measure, are of the form $d\mu(t) = f(t)dm(t)$ where f is continuous and $\{T_s f | s \in G\}$ spans a finite dimensional linear space of functions. More generally everyone of these measures is such that $\{T_s \mu | s \in G\}$ spans a finite dimensional linear space of measures. It seems natural then to study the questions discussed in the previous section not for measures which act invariantly but for ones which act almost invariantly in the sense that the translates of the functionals which the measures define span finite dimensional spaces. In doing this one hopes to obtain more complete answers to the questions posed in the first section in a more general setting than considered there. As we shall see this hope is only partially fulfilled.

In the preceding discussion the fact that G is abelian plays no essential role and in the following development we shall generally require that G be a locally compact topological (LC) group. First we must make some definitions.

If μ is a measure on G or h is a function on G then $T_s\mu$ and $T_s h$ are defined as before, while $T^s\mu$ and $T^s h$ are defined by $T^s\mu(E) = \mu(sE)$ and $T^s h(t) = h(st)$. m and m' will denote, respectively, right and left invariant Haar measure on the group G , that is $T_s m = m$ and $T^s m' = m'$ for all $s \in G$. The space of all complex valued regular Borel measures on G which are finite on compact sets will be denoted by $V(G)$, while $M(G)$ is the subspace of $V(G)$ consisting of those measures with finite total mass. A linear subspace X of $C_0(G)$ is said to be <u>right (left) translation invariant</u> if $T_s h \in X$ $(T^s h \in X)$ whenever $h \in X$ for all $s \in G$. X is <u>translation invariant</u> if it is both right and left translation invariant. The space of all complex valued linear functionals on X will be denoted by $L(X)$. We shall always consider this space as a topological linear space with the topology given by pointwise convergence on X , that is, a net of functionals $\{F_\alpha\} \subset L(X)$ converges to $F \in L(X)$ if and only if $\lim_\alpha F_\alpha(h) = F(h)$ for each $h \in X$. It should be noted that the elements of $L(X)$ are not necessarily continuous.

If $\mu \in V(G)$ and X is a right translation invariant subspace of $C_0(G)$ then μ is said to <u>act right almost invariantly on</u> X if

$$\text{i)} \quad \int_G |h(t)|\,d|\mu|(t) < \infty \qquad\qquad (h \in X)$$

and

$$\begin{aligned}\text{ii)} \quad \int_G T_{s^{-1}} h(t)\,d\mu(t) &= \int_G h(t)\,dT_s\mu(t)\\ &= \sum_{i=1}^{n} \alpha_i(s) \int_G h(t)\,dT_{s_i}\mu(t)\\ &= \sum_{i=1}^{n} \alpha_i(s) \int_G T_{s_i^{-1}} h(t)\,d\mu(t)\end{aligned}$$

for all $s \in G$ and $h \in X$ where $s_1, s_2, \dots, s_n$ are certain fixed elements of G.

If for each $s \in G$ we define the funtional $F_s \in L(X)$ by

$$F_s(h) = \int_G T_{s^{-1}}h(t)d\mu(t) = \int_G h(t)dT_s\mu(t) \qquad (h \in X)$$

then μ acts right almost invariantly on X precisely if $\{F_s | s \in G\}$ spans a finite dimensional subspace of $L(X)$. In this case we can write

$$F_s = \sum_{i=1}^{n} \alpha_i(s)F_{s_i} \qquad (s \in G),$$

and assume without loss of generality that $F_{s_1}, F_{s_2}, \dots, F_{s_n}$ are linearly independent elements of $L(X)$.

A measure $\mu \in V(G)$ is said to be <u>right almost invariant</u> if $\{T_s\mu | s \in G\}$ spans a finite dimensional space of measure. This is equivalent by the regularity of μ to the requirement that μ act right almost invariantly on $C_c(G)$.

Measureswhich act left almost invariantly or are left almost invariant are defined in the obvious manner. For convenience we shall concentrate our attention primarily on the "right" situation. It will be apparent what the appropriate translation of our results should be to cover the "left" case. The most important difference is the substitution of left invariant Haar measure for right invariant Haar measure. Obviously for abelian groups there is no difference between the two concepts. When we speak in generalities we shall often drop the adjectives "right" and "left" so as to include both possibilities.

Evidently any measure which acts invariantly also acts almost invariantly, and the measures considered in the examples of the previous section are all almost invariant measures.

In analogy with the discussion of the first section we pose the following questions. If μ acts right almost invariantly on X then :

I') What can be said about μ ? In particular when is μ a right almost invariant measure ?

II') What can be said about the linear functional F_e ?

We now also have a third question of some interest.

III') What, precisely, is the nature of a right almost invariant measure ?

In the following sections we shall obtain partial answers to these questions.

REMARK. Obviously if μ is a right almost invariant measure then μ acts right almost invariantly on any right translation invariant subspace $X \subset C_o(G)$ for which $\int_G |h(t)|d|\mu|(t) < \infty$ for all $h \in X$.

3. The main theorem for measures which act almost invariantly and the characterization of almost invariant measures. The main theorem of this section is the first step in the direction of an answer to question II' and also can be considered as a generalization of Theorem 2. An immediate corollary of the theorem will be a characterization of almost invariant measures. Since this theorem is so central to our development we shall give a brief outline of the proof. Detailed proofs for this and other results mentioned below can be found in [3,4] .

THEOREM 3. Let G be a LC group, X a right translation invariant subspace of $C_o(G)$ and $\mu \in V(G)$. If μ acts right almost invariantly on X then there exists a continuous function f

such that

$$F_e(h) = \int_G h(t)d\mu(t) = \int_G h(t)f(t)dm(t) \qquad (h \in X) .$$

OUTLINE OF PROOF. Assume that $F_s = \sum_{i=1}^{n} \alpha_i(s)F_{s_i}$ for each $s \in G$ and that $F_{s_1}, F_{s_2}, \dots, F_{s_n}$ are linearly independent in $L(X)$. Let C be the finite dimensional subspace of $L(X)$ spanned by $F_{s_1}, F_{s_2}, \dots, F_{s_n}$.

Step 1. First we need to show that $\alpha_1, \alpha_2, \dots, \alpha_n$ are continuous functions on G. If $X \subset C_c(G)$ then this follows quite readily from the continuity of each of the functions $s \longrightarrow F_s(h) = \int_G T_{s^{-1}}h(t)d\mu(t)$, $h \in X$, the continuity of the projections onto the i-th co-ordinate in C and the finite dimensionality of C. However if $X \not\subset C_c(G)$ then the continuity of $s \longrightarrow F_s(h)$ is no longer a priori evident as for arbitrary $\mu \in V(G)$ and $h \in C_o(G)$ the mapping $s \longrightarrow F_s(h)$ may fail to be continuous. In this latter case one constructs a certain finite dimensional Haar measurable - and hence continuous - representation of the group G such that the α_i can be expressed as finite linear combinations of the continuous entries in the matrices of the representation. The representation is essentially that given by the translation operators acting on C. Of course this argument also works when $X \subset C_c(G)$.

Step 2. For each $g \in C_c(G)$ define

$$F_g = \int_G g(s)F_s dm(s) = \sum_{i=1}^{n} \left(\int_G g(s)\alpha_i(s)dm(s)\right) F_{s_i} ,$$

and let $B = \{F_g | g \in C_c(G)\}$. Then B is a closed linear subspace of C. If $\{g_\beta\} \subset C_c(G)$ is a net of functions such that i) $g_\beta \geq 0$, all β, ii) $\int_G g_\beta(t)dm(t) = 1$, all β, and

iii) for any open symmetric neighbourhood U of the identity in G that is a β_0 such that if $\beta > \beta_0$ then the support of g_β is contained in U, that is, $\{g_\beta\}$ is a <u>compact approximate identity</u>, then from the continuity of the α_i we conclude that $\lim_\beta F_{g_\beta} = F_e$. Hence $F_e \in B$.

Step 3. Thus there exists some $k \in C_c(G)$ such that $F_e = F_k$. An application of Fubini's theorem reveals that

$$F_e(h) = \int_G h(t)f(t)d\mu(t) \qquad\qquad (h \in X)$$

where $f(t) = \Delta_\ell(t^{-1}) \int_G k(t^{-1}s)d\mu(s)$ and Δ_ℓ is the left modular function for G.

This completes the proof.

Let us denote by $FDT(G)$ the space of all complex valued continuous functions f on G such that $\{T_s f | s \in G\}$ spans a finite dimensional space. In this definition it is irrelevant whether we speak of functions whose left or right translates span finite dimensional spaces. This is true because of the following lemma whose proof is elementary.

LEMMA. Let G be a LC group and f a function defined on G. Then the following are equivalent:

i) $\{T_s f | s \in G\}$ spans a finite dimensional space.

ii) $\{T^s f | s \in G\}$ spans a finite dimensional space.

By $D(G)$ we denote the space of all linear combinations of products of complex valued continuous functions on G which are either additive or multiplicative, that is, functions f such that either

$$f(st) = f(s) + f(t) \quad \text{or} \quad f(st) = f(s)f(t) .$$

If G is an abelian group then $FDT(G) = D(G)$ [9, p. 25]

An immediate corollary of Theorem 3 is the following result.

COROLLARY. Let G be a LC group and $\mu \in V(G)$. Then the following are equivalent:

i) μ is a right almost invariant measure.

ii) There is an $f \in FDT(G)$ such that $d\mu(t) = f(t)dm(t)$.

If G is abelian then $f \in D(G)$.

In particular this says that right almost invariant measures are absolutely continuous with respect to right invariant Haar measure.

Since the left and right modular functions of a LC group belong to $FDT(G)$ we obtain also the next corollary.

COROLLARY. Let G be a LC group and $\mu \in V(G)$. Then the following are equivalent:

i) μ is right almost invariant.

ii) μ is left almost invariant.

Thus we can legitimately speak of almost invariant measures without the modifiers right and left.

A final corollary of the characterization of almost invariant measures is the next result.

COROLLARY. Let G be a LC group, $\mu \in V(G)$, $\mu \neq 0$. If μ is singular with respect to Haar measure on G then $\{T_s\mu \mid s \in G\}$ and $\{T^s\mu \mid s \in G\}$ span infinite dimensional subspaces of $V(G)$.

REMARKS. a) Clearly the function f provided by Theorem 3 is not, in general, unique.

b) If $G = R^k$, $k > 0$, it is easy to see that f can be chosen to be infinitely differentiable. Also in this case if μ is an almost invariant measure then μ is of the form

$$d\mu(t) = \sum_{j=1}^{r} P_j(t)\exp(e_j,t)dm(t)$$

where P_j are polynomials with complex coefficients, the b_j are k-vectors of complex numbers and $t = (x_1, x_2, \ldots, x_k) \in R^k$.

c) If μ is almost invariant and, for example, $T_s\mu = \sum_{i=1}^{n} \alpha_i(s)T_{s_i}\mu$, $s \in G$, then $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent elements of FDT(G) and we may write $f = \sum_{i=1}^{n} f(s_i)\alpha_i$. Furthermore the dimensions of the spaces spanned by the $\{T_s\mu | s \in G\}$ and $\{T_sf | s \in G\}$ are equal.

d) If G is compact and abelian then it is easily seen that FDT(G) consists precisely of the trigonometric polynomials on G. More generally whenever G is abelian the bounded elements of FDT(G) are exactly the trigonometric polynomials on G.

e) It is perhaps worth while to point out that the proof of Theorem 3 establishes slightly more than asserted. Namely, if $F_e \in L(X)$ and $F_s \in L(X)$ is defined by $F_s(h) = F_e(T_{s^{-1}}h)$ then whenever the mapping $s \to F_s$ is continuous from G to $L(X)$ and $\{F_s | s \in G\}$ spans a finite dimensional subspace of $L(X)$ we can conclude that there exists some $k \in C_c(G)$ such that

$$F_e = \int_G k(s)F_s dm(s) .$$

One can also replace m by any non-negative regular Borel measure on G.

4. Uniqueness theorems. Let us now turn our attention to question I' . As before the almost invariance of a measure μ cannot be deduced from the fact that μ acts almost invariantly on some invariant subspace X. Nor does acting almost invariantly imply the absolute continuity of the measure μ . For example let G be any infinite compact abelian group, X the space spanned by any continuous character $(\cdot,\gamma)$ on G which is not identically one and suppose μ is the measure with unit mass concentrated at the identity. Then μ is neither almost invariant nor absolutely continuous but it does act almost invariantly on X since

$$\int_G (t,\gamma)\,dT_s\mu(t) = (s^{-1},\gamma)\int_G (t,\gamma)\,d\mu(t) \qquad (s \in G) .$$

The analog of Theorem 1 is however valid.

THEOREM 4. Let G be a LC group, X a right translation invariant subalgebra of $C_o(G)$ which is dense in $C_o(G)$ and $\mu \in V(G)$. If μ acts right almost invariantly on X then μ is an almost invariant measure.

The proof of this result is essentially the same as for Theorem 1 given in (2).

When G is compact the functionals F_s are continuous and so the implication of the preceding theorem remains valid under the assumption that X is only a dense translation invariant subspace of $C_o(G) = C_c(G) = C(G)$. If G is noncompact it is also possible to replace the hypothesis that X is a subalgebra by another condition.

THEOREM 5. Let G be a LC group, X a right translation invariant subspace of $C_o(G)$ which contains a compact approximate identity and $\mu \in V(G)$. If μ acts right almost invariantly on X then μ is an almost invariant measure.

The proof consists in showing that the mapping $\phi : T_s\mu \to F_s$ is a linear injective and surjective mapping from the space spanned by $\{T_s\mu \mid s \in G\}$ to the finite dimensional subspace of $L(X)$ spanned by $\{F_s \mid s \in G\}$.

REMARK. The hypotheses of Theorem 5 do not however circumvent the density assumption in Theorem 4 since any right translation invariant subspace of $C_o(G)$ which contains a compact approximate identity is dense in $C_o(G)$.

5. <u>More on measures which act almost invariantly.</u> As observed previously if a measure μ acts invariantly or almost invariantly then we cannot in general conclude that μ is itself an invariant or almost invariant measure. We do however know from Theorem 3 that if μ acts right almost invariantly on X then there always exists a continuous function f such that

$$\int_G h(t)d\mu(t) = \int_G h(t)f(t)dm(t) \qquad (h \in X) .$$

In certain situations f may be chosen so that it belongs to $FDT(G)$, that is, so that the action of μ as a functional on X is the same as that for a functional defined by some almost invariant measure. For example this was the case in Theorem 2 cited earlier. Another instance is provided by the following result from (2).

THEOREM 6. Let X be a translation invariant subspace of $C_c(R)$ and suppose $\mu \in V(G)$ acts invariantly on X. Let p be the smallest non-negative integer such that

$$\int_R g(t)t^p dm(t) \neq 0$$

for some $g \in X$. Then there is a constant c such that

$$\int_R h(t)d\mu(t) = c\int_R h(t)t^p dm(t) \qquad (h \in X) .$$

The constant $c = \int_R g(t)d\mu(t)/\int_R g(t)t^p dm(t)$.

Thus in this context we can find an almost invariant measure which considered as a functional is identical with the functional μ which acts invariantly. It should be noted that the dimension of the space spanned by the translates of this almost invariant measure is in general strictly greater than one, the dimension of the space spanned by the translates of the functional defined by μ. Moreover the almost invariant measure is not unique since it is evident that $d\nu(t) = (t^p + t^k)dm(t)$, $1 \leq k < p$, will also work.

Our attention in this section will thus be focused on question II'. In particular given a measure μ which acts almost invariantly we shall seek conditions which insure the existence of an almost invariant measure which coincides with μ as an element of $L(X)$. We shall first discuss some results for arbitrary LC groups. These are definitely unsatisfactory but do generalize Theorem 2. For compact groups we shall see that an almost invariant measure with the desired properties always exists.

To insure that the proofs in the non-compact case are valid we must however consider translation invariant subspaces X rather than subspaces which are either right or left invariant. In this context if the answer to the preceding question is in the affirmative then the notions of acting right or left almost invariantly are equivalent. We state this result as the next theorem.

THEOREM 7. Let G be a LC group, X a translation invariant subspace of $C_0(G)$ and $\mu \in V(G)$. If μ acts right (left) almost invariantly on X and there exists an almost invariant measure ν such that

$$\int_G h(t)d\mu(t) = \int_G h(t)d\nu(t) \qquad (h \in X),$$

then μ acts left (right) almost invariantly on X.

Consequently we shall for the moment restrict our attention to measures which act almost invariantly, that is, act right and left almost invariantly on a translation invariant space X.

Theorem 2 is an immediate consequence of the next result.

THEOREM 8. Let G be a LC group, X a translation invariant subspace of $C_0(G)$. Suppose $\mu \in V(G)$ acts almost invariantly on X and that

$$\int_G T_{s^{-1}}h(t)d\mu(t) = \sum_{i=1}^{n} \alpha_i(s)\int_G T_{s_i^{-1}}h(t)d\mu(t) \qquad (s \in G, h \in X).$$

If there exists a function $g \in X$ such that :

i) $\int_G |g(t)|dm(t) < \infty$

ii) $\int_G |g(t)\alpha_i(t)|dm(t) < \infty$, $\qquad i = 1, 2, \ldots, n$

iii) $\int_G g(t)\alpha_i(t)dm(t) = \alpha_i(e)$, $\qquad i = 1, 2, \ldots, n$

then there exists an almost invariant measure ν such that

$$\int_G h(t)d\mu(t) = \int_G h(t)d\nu(t) \qquad (h \in X).$$

The hypotheses of the theorem are stated with reference to the fact that μ acts right almost invariantly. Of course similar assumptions can be made utilizing the fact that μ acts left almost invariantly.

If one recalls the proof of Theorem 3 then we know that if μ acts almost invariantly on X then there exists some $k \in C_c(G)$ such that

$$F_e(h) = \int_G k(s)F_s(h)dm(s) \qquad (h \in X) .$$

Combining this recollection with the previous theorem enables one to easily prove the following corollary.

COROLLARY. Let G be a LC group, X a translation invariant subspace of $C_o(G)$. If $\mu \in V(G)$ acts almost invariantly on X and $k \in X$ then there exists an almost invariant measure ν such that

$$\int_G h(t)d\mu(t) = \int_G h(t)d\nu(t) \qquad (h \in X) .$$

Under the hypotheses of the corollary k has all the properties of the function g in the preceding theorem.

REMARK. It is not clear whether the rather special assumption in Theorem 8 can be removed. From the example discussed at the end of the first section it is however evident that if one requires the dimension of the space spanned by $\{F_s | s \in G\}$ in $L(X)$ to be equal to the dimension of the space spanned by $\{T_s\nu | s \in G\}$ in $V(G)$ where ν is an almost invariant measure for which

$$F_e(h) = \int_G h(t)d\nu(t) \qquad (h \in X) ,$$

then some sort of restrictions must be placed on X .

Clearly these results leave much to be desired. The situation for compact groups is much more satisfactory.

THEOREM 9. Let G be a compact group, X a right translation invariant subspace of $C(G)$, and suppose $\mu \in M(G)$ acts right almost invariantly on X. Then there exists an almost invariant measure ν such that

$$\int_G h(t)d\mu(t) = \int_G h(t)d\nu(t) \qquad (h \in X).$$

OUTLINE OF PROOF. Let $\{g^\gamma\}_{\gamma\in\Gamma}$ be a complete family of finite dimensional continuous irreducible inequivalent unitary representations of G. For each $t \in G$, $g^\gamma(t) = (g^\gamma_{ij}(t))$ is an $r(\gamma)\times r(\gamma)$ unitary matrix and the functions g^γ_{ij} belong to $FDT(G)$. Since G is compact we may assume that X is closed. Let Δ be the collection of all g^γ_{ij} which belong to X and such that

$$\int_G g^\gamma_{ij}(t)d\mu(t) \neq 0 .$$

Then either $\Delta = \phi$, in which case the theorem is trivially true with $\nu = 0$, or Δ is finite. Denoting the distinct elements of Δ as $g^{\gamma_k}_{ij}$, $i = 1, 2, \ldots, m(k)$, $j = 1, 2, \ldots, n(k)$, $k = 1, 2, \ldots, p$ we define

$$f(t) = \sum_{k=1}^{p}\sum_{i=1}^{m(k)}\sum_{j=1}^{n(k)} r(\gamma_k)\left(\int_G g^{\gamma_k}_{ij}(s)d\mu(s)\right)\overline{g^{\gamma_k}_{ij}}(t) .$$

Using the orthogonality relations of the $\{g^\gamma_{ij}\}$(10, p. 73-74)it is easily seen that $d\nu(t) = f(t)dm(t)$ is the desired almost invariant measure.

REMARKS. a) If G is abelian then f is a linear combination of the continuous characters which are common to X and the support of the Fourier-Stieltjes transform of μ.

b) In the non-abelian case one cannot, in general, obtain f

as a linear combination of the characters of the representations g^γ. For example let g^γ be a representation and let g^γ_{ij} be any element such that $i \neq j$. Set X equal to the closed linear span of $\{T_s \overline{g^\gamma_{ij}} | s \in G\}$ and $d\mu(t) = g^\gamma_{ij}(t)dm(t)$. μ is an almost invariant measure on G and so acts right almost invariantly on X. If $f = \sum_{k=1}^{n} c_k X_k = \sum_{k=1}^{n} c_k \sum_{m=1}^{r(\gamma_k)} g^{\gamma_k}_{mm}$ then, since $i \neq j$, the orthogonality relations reveal that $\int_G g^\gamma_{ij}(t)f(t)dm(t) = 0$ but $\int_G \overline{g^\gamma_{ij}}(t)d\mu(t) = 1/r(\gamma) \neq 0$.

6. Generalizations and related results. In the previous discussions we have restricted our attention to measures μ on a LC group G such that μ acts right almost invariantly on X. We have already commented on the role of the adjectives "right" and "left" for the preceding results. The assumption that $\{F_s | s \in G\}$ - or for almost invariant measures that $\{T_s\mu | s \in G\}$ - spans a finite dimensional space is mainly one of convenience, and the validity of a number of our results does not depend on the fact that we allow translation by all elements of G. Indeed we can obtain many of the same theorems by considering only translation by elements in some subset of positive Haar measure. To be specific, let X be a right translation invariant subspace of $C_0(G)$ and U a Borel subset of G such that $m(U) > 0$. We say μ acts U-right almost invariantly on X if

$$\int_G T_{s^{-1}}h(t)d\mu(t) = \sum_{i=1}^{n} \alpha_i(s)\int_G T_{s_i^{-1}}h(t)d\mu(t) \qquad (s \in U, h \in X).$$

If $X = C_c(G)$ the μ is a U-right almost invariant measure.

With essentially the same proofs as before one can prove the following theorem and corollary.

THEOREM 10. Let G be a LC group, X a right translation invariant subspace of $C_o(G)$ and $\mu \in V(G)$. If U is a set of positive Haar measure such that μ acts U-right almost invariantly on X then there exists a continuous function f on G such that

$$\int_G h(t)d\mu(t) = \int_G h(t)f(t)dm(t) \qquad (h \in X).$$

COROLLARY. Let G be a LC group, $\mu \in V(G)$ and U a set of positive Haar measure. Then the following are equivalent:

i) μ is a U-right almost invariant measure.

ii) There is a continuous function f on G such that $d\mu(t) = f(t)dm(t)$ and $\{T_s f | s \in U\}$ spans a finite dimensional space of functions.

Counterparts of the uniqueness theorems and some other results discussed above can also be proved. If G is connected then the notions of acting U-right almost invariantly and right almost invariantly are equivalent. The reader is referred to (3) for details.

For Euclidean groups some results related to the problem as described in the preceding paragraphs have been obtained by Anselone and Korevaar (1). A special case of one of their theorems was proved independently by Larsen (5). Their theorem stated in terms of the group R is given below.

THEOREM 11. Let μ be a Schwartz distribution on R and $s, t \in R$ be such that s/t is irrational. Then the following are equivalent:

i) $\mu \in D(R)$.

ii) μ is contained in a finite dimensional space W of Schwartz distributions on R, and if $\nu \in W$ then $T_s\nu \in W$ and $T_t\nu \in W$.

The condition on s and t implies that W is invariant under translation by elements of the dense subgroup of R generated by s and t, and hence under translation by all of R.

It should be noted that the assumptions on μ made in this context are not precisely analogous to those considered previously. In the latter context we assume that μ belongs to a finite dimensional space which is invariant under certain translations. While in the former we assume that the collection of translates of μ by elements of a certain set of positive Haar measure U spans a finite dimensional space W. If G is connected it follows from this that W is invariant under translation by elements of G. However if G is not connected then W may not even be invariant under translation by elements of U. For example let $G = Z$, the group of integers under addition, $U = \{-1, 0, 1\}$ and define $\mu \in V(G)$ by $\mu(k) = 1$ if $k = 4n$, $\mu(k) = 0$ otherwise. Then the set $\{T_s\mu | s \in U\} = \{T_{-1}\mu, \mu, T_1\mu\}$ spans a three dimensional space W. However the space spanned by $\{T_s\nu | \nu \in W, s \in U\} = \{T_{-2}\mu, T_1\mu\ \mu, T_1\mu, T_2\mu\}$ is four dimensional.

Finally it is possible to consider measures which act almost invariantly on translation invariant subspaces of $C_0(S)$ where S is a certain type of semi-group in G, in particular an open semi-group in G which contains the identity of G in its closure. One can then prove all the theorems we have discussed previously essentially by reducing the problem on semi-groups to a parallel one on groups. The conditions on the semi-groups are however necessary for the validity of the results. The interested reader is referred to (3) for more details.

REFERENCES

1. P.M. Anselone and J. Korevaar, Translation invariant subspaces of finite dimension, Proc. Amer. Math. Soc. 15(1964), 747-752.

2. M. Jerison and W. Rudin, Translation invariant functionals, Proc. Amer. Math. Soc. 13(1962), 417-423.

3. R. Larsen, Almost invariant measures, Proc. J. Math. 15(1965), 1295-1305.

4. ________, Measures which act almost invariantly (submitted).

5. ________, Distributions with finite dimensional translates (unpublished).

6. L. Loomis, An Introduction to Abstract Harmonic Analysis, Van Nostrand, Princeton, 1953.

7. M.A. Naimark, Normed Rings, Noordhof, Groningen, 1959.

8. W. Rudin, Fourier Analysis on Groups, Interscience Publishers, New York, 1962.

9 J. Stone, Exponential Polynomials on Commutative Semi-groups, Applied Mathematics and Statistics - Laboratories Technical Note No. 14, Stanford University, 1960.

10. A. Weil, L'Integration dans les groupes topologiques et ses applications, Hermann and Cie, Paris, 1953.